4.6. SPAN OF TRIGONOMETRIC SUMS IN WEIGHTED L^2 SPACES*

Let $\Delta = \Delta(\gamma)$ be an odd nondecreasing bounded function of γ on the line $\mathbb{R}$, let $Z(\Delta) = L^2(\mathbb{R}, d\Delta)$ and let $Z^T(\Delta)$ denote the closure in $Z(\Delta)$ of finite trigonometric sums $\Sigma c_j e^{i\gamma t_j}$ with $|t_j| \leqslant T$. It is readily checked that $Z^{T_1}(\Delta) \subset Z^{T_2}(\Delta)$ for $T_1 \leqslant T_2$ and that $\bigcup_{T>0} Z^T(\Delta)$ is dense in $Z(\Delta)$. Let

$$T_0(\Delta) = \inf\{T > 0 : \bar{Z}^T(\Delta) = Z(\Delta)\}$$

with the understanding that $T_0(\Delta) = \infty$ if the equality $Z^T(\Delta) = Z(\Delta)$ is never attained. The following three examples indicate the possibilities:

(1) if $\Delta(\gamma) = \int_0^\gamma (\xi^2+1)^{-1} d\xi$, then $T_0 = \infty$;

(2) if $\Delta(\gamma) = \int_0^\gamma e^{-|\xi|} d\xi$, then $T_0 = 0$;

(3) if Δ is a step function with jumps of height $1/(n^2 + 1)$ at every integer n, then $T_0 = \pi$.

Problem. Find formulas for T_0, or at least bounds on T, in terms of Δ.

Discussion. Let Δ' denote the Radon–Nikodym derivative of Δ with respect to Lebesgue measure. It then follows from a well-known theorem of Krein [1] that $T_0 = \infty$ as in example (1) if

$$\int_{-\infty}^{\infty} \frac{\log \Delta'(\gamma)}{\gamma^2+1} d\gamma > -\infty .$$

A partial converse due to Levinson–McKean implies that if Δ is absolutely continuous and if $\Delta'(\gamma)$ is a decreasing function of $|\gamma|$ and $\int_{-\infty}^{+\infty} \frac{\log \Delta'(\gamma)}{\gamma^2+1} d\gamma = -\infty$ [as in example (2)], then $T_0 = 0$.

A proof of the latter and a discussion of example (3) may be found in Sec. 4.8 of [2]. However, apart from some analogues for the case in which Δ is a step function with jumps at the integers, these two theorems seem to be the only general results available for computing T_0 directly from Δ. (There is an explicit formula for T_0 in terms of the solution to an inverse spectral problem, but this is of little practical value because the computations involved are typically not manageable.)

The problem of finding T_0 can also be formulated in the language of Fourier transforms since $Z^T(\Delta)$ is a proper subspace of $Z(\Delta)$ if and only if there exists a nonzero function $f \in Z(\Delta)$ such that

$$\tilde{f}(t) = \int_{-\infty}^{+\infty} e^{i\gamma t} f(\gamma)\, d\Delta(\gamma) = 0$$

for $|t| \leqslant T$. Thus

$$T_0 = \inf\{T > 0 : \tilde{f}(t) = 0 \text{ for } |t| \leqslant T \Rightarrow f = 0 \text{ in } Z(\Delta)\}.$$

Special cases of the problem in this formulation have been studied by Levinson [3] and Mandelbrojt [4] and a host of later authors. For an up-to-date survey of related results in the special case that Δ is a step function see [5]. The basic problem can also be formulated in

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$L^p(\mathbb{R}, d\Delta)$ for $1 \leqslant p \leqslant \infty$. A number of results for the case $p = \infty$ have been obtained by Koosis [6-8].

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